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American options analyzed differently

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Abstract

In this note we analyze in a discrete-time context and with a finite outcome space American options starting with the idea that every tradable should be a martingale under a certain measure. We believe that in this way American options become more understandable to people with a good working knowledge of European options and a basic understanding of stochastic processes in discrete-time. Invariably we will assume that the underlying market is complete.

Introduction

It takes some time to understand that in a complete market, say, in a Black-Scholes context, a European option, say a call or a put, has a price process in which the drift of the underlying asset is not explicit. It also takes time that, modulo technical conditions, tradables are martingales under an equivalent measure. But gradually the student becomes confident and really thinks that she understands the basics of the pricing of derivatives. But then very often comes a (minor) stock. Although in nonmathematical terms it is perfectly clear what an American option is, it is nevertheless to some of the students a surprise that in many treatments of the mathematics of American options its price processes are defined in such a way that these processes are supermartingales and not martingales. So it seems that the paradigm that tradables are martingales is not correct any more in this setting. Of course, pretty soon they more or less understand that when you optimally exercise, then the price process until and including the exercise date is a martingale.

In this note we take a slightly different route, by not abandoning for a moment martingales. It is quite well possible that this different approach is of some help to novices in the area of American options. In any case the author of this paper finds his route intellectually quite appealing.

In the rest of this note we presume familiarity with the basic results of stochastic processes in discrete time and with the theory of pricing European options in discrete time. All underlying markets will be complete, and we will also assume that our probability space has a finite outcome space. This way our philosophy will not be unnecessarily obscured by difficult technicalities.

Setting the stage

Our probability space is (Ω, \mathcal{F}, P) , where Ω has finitely many elements. Our time axis is equal to

$$\widehat{T} := \{0, 1, 2, \dots, T\},$$

where T is a given integer greater than zero. Date “0” has the interpretation of present, hence all the other dates are in the future. In order to model the flow of information we introduce a filtration $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \dots \mathcal{F}_{T-1} \subseteq \mathcal{F}_T$, where it is assumed that \mathcal{F}_0 is trivial, i.e., $\mathcal{F}_0 := \{\Omega, \emptyset\}$, and that $\mathcal{F}_T = \mathcal{F}$.

Our hypothetical market is modelled by a stochastic process:

$$\begin{aligned} S &: \widehat{T} \times \Omega \rightarrow \mathbf{R}^{d+1}, \text{ where} \\ S &:= (S^0, S^1, \dots, S^d), \text{ where} \\ d &\text{ is an integer greater than zero.} \end{aligned}$$

We further assume:

$$\forall i, \forall t \in \widehat{T}, \forall \omega \in \Omega : S^i(t, \omega) > 0.$$

$$\forall t \in \widehat{T}, \forall \omega \in \Omega : S^0(t, \omega) = 1.$$

Therefore S^0 fulfills the role of bank account with zero interest. We also assume:

$$\forall i, \forall t \in \widehat{T} : S^i(t, \cdot) \in m\mathcal{F}_t,$$

where $m\mathcal{F}_t$ is by definition the set of random variables measurable with respect to \mathcal{F}_t .

Therefore we may view, $\forall i > 0$, S^i as the discounted price process of a tradable asset i .

We further assume that there is a unique probability measure Q equivalent to P such that:

$$\forall i, \forall t, \forall s : E^Q(S^i(t) | \mathcal{F}_s) = S^i(s), \text{ when } t \geq s.$$

In words: all discounted price processes are Q -martingales with respect to the given filtration $\{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$. Hence our market is arbitrage-free and complete.

In the sequel we denote by the capital letter A (for American) a fixed stochastic process

$$\begin{aligned} A & : \widehat{T} \times \Omega \rightarrow \mathbf{R}, \text{ with} \\ \forall t \in \widehat{T} & : A(t, \cdot) \in m\mathcal{F}_t \\ \forall t \in \widehat{T}, \forall \omega \in \Omega & : A(t, \omega) \geq 0. \end{aligned}$$

The interpretation of A is as follows: when the holder of the American option exercises at $t \in \widehat{T}$ when nature has chosen $\omega \in \Omega$ the net return to the holder is $A(t, \omega)$. This means that in the sequel we have to speak about stopping times. Recall that $\tau : \Omega \rightarrow \widehat{T} \cup \{+\infty\}$ is a stopping time if and only if

$$\{\omega \in \Omega : \tau(\omega) = t\} =: (\tau = t) \in m\mathcal{F}_t, \forall t \in \widehat{T}.$$

Invariably stopping times will be denoted by the letter “ τ ” with or without subscripts, hats or tildes.

And also invariably all our stopping times in the sequel have their images in \widehat{T} (you *have* to exercise your American option!).

Also recall that for every stopping time τ :

$$\mathcal{F}_\tau := \{F \in \mathcal{F} : F \cap (\tau = t) \in \mathcal{F}_t, \forall t \in \widehat{T} \cup \{+\infty\}\},$$

where by definition $\mathcal{F}_\infty = \mathcal{F}$. Notice that \mathcal{F}_τ is a σ -algebra as well.

A final piece of notation: When $H \subseteq \Omega$ is a set, then $\chi(H)$ is the indicator function of H , meaning that: $\forall \omega \in \Omega$:

$$\begin{aligned} \chi(H)(\omega) &= 1 \quad \text{whenever } \omega \in H \\ \chi(H)(\omega) &= 0 \quad \text{whenever } \omega \notin H. \end{aligned}$$

American options

Before we start with a more formal analysis of American option we notice that for a given stopping time τ and a given nonnegative random variable $X \in m\mathcal{F}_\tau$, the pair (τ, X) may

be viewed as a European option that has to be exercised at time τ , where τ is random!

Its the price process therefore has to be the mapping

$$\widehat{T} \ni t \rightarrow E^Q(X \mid \mathcal{F}_t),$$

a Q -martingale, where $E^Q(X \mid \mathcal{F}_t)(\omega)$ is the price of this product when $t \leq \tau(\omega)$, and where one may imagine at time $\tau(\omega)$ to invest the proceeds $X(\tau(\omega))$ in the bank account.

In the sequel the following result is very important.

Theorem 1. The following two statements are true.

- a. $\forall t \in \widehat{T}, \exists \hat{\tau}$ with $\hat{\tau} \geq t$ such that: $\forall \omega \in \Omega, \forall \tau$ with $\tau \geq t$:

$$E^Q(A(\hat{\tau}) \mid \mathcal{F}_t)(\omega) \geq E^Q(A(\tau) \mid \mathcal{F}_t)(\omega).$$

 With: $P_A(t) := E^Q(A(\hat{\tau}) \mid \mathcal{F}_t)$, where $\hat{\tau}$ is as above:
- b. $P_A(T) = A(T)$

$$\forall \omega \in \Omega, \forall t \geq 1, t \in \widehat{T} : P_A(t-1)(\omega) =$$

$$= \max\{A(t-1, \omega), E^Q(P_A(t) \mid \mathcal{F}_{t-1})(\omega)\}.$$

Before we give a proof of this result we make the following comments.

1. Statement a. is trivial for $t = 0$, as Ω has a finite number of elements and hence there are only finitely many stopping times. *However*, that for $t \geq 1$ there is (*uni-*
formly over Ω) a best stopping time with values greater than or equal to t is in my opinion not trivial, hence part a. really needs a proof.
2. Very often one *defines* the price process of A by means of the recursion under b. However, P_A is in general not a martingale, it is a supermartingale.
 The intuition behind recursion b. is pretty clear: When the American option is alive at T its value obviously is equal to $A(T)$. Now assume that the option is still alive at time $t-1$, then you have the choice between immediately exercising it, or waiting at least until t . But at t its price is equal to $P_A(t)$, hence the recursion formula intuitively is correct. The disadvantage is that we have to use expressions like: “the option is alive”, where this notion has not yet a proper formal interpretation. So far, so good. We continue with:

Proof of theorem 1. Let us denote by $\Sigma(t)$ the set of all stopping times τ with $\tau(\omega) \geq t$, $\forall \omega \in \Omega$. Notice, that trivially a. and b. are true for $t = T$. The rest of the proof goes by an induction argument and runs as follows.

We assume that for all $s = \{T, T-1, \dots, t\}$ there is a stopping time $\hat{\tau}(s) \in \Sigma(s)$ such that:

$$\forall \omega \in \Omega, \forall \tau \in \Sigma(s) : E^Q(A(\hat{\tau}(s)) \mid \mathcal{F}_s)(\omega) \geq E^Q(A(\tau) \mid \mathcal{F}_s)(\omega).$$

We also assume that, with $P_A(s) := E^Q(A(\hat{\tau}(s)) \mid \mathcal{F}_s)$, $\forall s \in \{T, T-1, \dots, t\}$:

$$P_A(s)(\omega) = \max\{A(s, \omega), E^Q(P_A(s+1) \mid \mathcal{F}_s)(\omega)\}, \forall \omega \in \Omega,$$

at least when $s \neq T$.

We now take an arbitrary $\tau \in \Sigma(t-1)$ and we define $\tau(t) := \tau \chi(\tau \geq t) + t \chi(\tau = t-1)$.

It is easy to see that $\tau(t) \in \Sigma(t)$. By our induction assumption we have:

$$\begin{aligned} E^Q(A(\tau(t)) \mid \mathcal{F}_t) &\geq P_A(t), \text{ and hence} \\ E^Q(\chi(\tau \geq t)A(\tau(t)) \mid \mathcal{F}_t) &\leq \chi(\tau \geq t)P - A(t) \leq P_A(t), \text{ and hence} \\ E^Q(\chi(\tau \geq t)A(\tau(t)) \mid \mathcal{F}_{t-1}) &\leq E^Q(P_A(t) \mid \mathcal{F}_{t-1}), \text{ and hence} \\ \chi(\tau \geq t)E^Q(A(\tau(t)) \mid \mathcal{F}_{t-1}) &\leq E^Q(P_A(t) \mid \mathcal{F}_{t-1}), \text{ and hence} \\ \chi(\tau \geq t)E^Q(A(\tau) \mid \mathcal{F}_{t-1}) &\leq E^Q(P_A(t) \mid \mathcal{F}_{t-1}). \end{aligned} \tag{1}$$

As $E^Q(A(\tau) \mid \mathcal{F}_{t-1}) = \chi(\tau = t-1)A(t-1) + \chi(\tau \geq t)E^Q(A(\tau) \mid \mathcal{F}_{t-1})$ statement (1) implies that:

$$E^Q(A(\tau) \mid \mathcal{F}_{t-1}) \leq \max\{A(t-1), E^Q(P_A(t) \mid \mathcal{F}_{t-1})\}. \tag{2}$$

We now define the random variable η as follows

$$\begin{aligned} \eta := (t-1)\chi(A(t-1) \geq E^Q(P_A(t) \mid \mathcal{F}_{t-1})) + \hat{\tau}(t)\chi(A(t-1) < E^Q(P_A(t) \mid \mathcal{F}_{t-1})). \end{aligned}$$

By construction we have that $\eta \in \Sigma(t-1)$, and also that

$$\begin{aligned} E^Q(\eta \mid \mathcal{F}_{t-1}) &= \max\{A(t-1), E^Q(\hat{\tau}(t) \mid \mathcal{F}_{t-1})\} \\ &= \max\{A(t-1), E^Q(P_A(t) \mid \mathcal{F}_{t-1})\}, \end{aligned}$$

because of the definition of $P_A(t)$ and $\hat{\tau}(t)$.

Therefore we are done with the proof of our first theorem. \square

Notice that along the way we also proved the following result.

Theorem 2. Inductively we define the following stopping times. $\hat{\tau}(T) := T$ and, for $t < T$ we define:

$$\begin{aligned} \hat{\tau}(t) &= t, \quad \text{whenever } A(t) \geq E^Q(P_A(t+1) \mid \mathcal{F}_t) \\ \hat{\tau}(t) &= \hat{\tau}(t+1), \quad \text{whenever } A(t) < E^Q(P_A(t+1) \mid \mathcal{F}_t). \end{aligned}$$

Then we have for every $t \in \widehat{T}$ the following:

1. $E^Q(A(\hat{\tau}(t)) \mid \mathcal{F}_t) \geq E^Q(A(\tau) \mid \mathcal{F}_t), \forall \tau \in \Sigma(t).$
2. $E^Q(A(\hat{\tau}(t)) \mid \mathcal{F}_t) = P_A(t).$

This means, that we have a *constructive* procedure to find optimal stopping times. Later on we will show that this way we do not necessarily find all optimal stopping times.

Our first pricing results reads as follows.

Theorem 3. If the American option has a price at $t = 0$ this price has to be $P_A(0)$.

Proof: As our economy is complete we may assume without loss of generality the existence of European options $A(\tau)$ for every $\tau \in \Sigma(0)$. By $A(\tau)$ we mean the European option that has to be exercised at time τ and gives a net payment of $A(\tau)$.

It is easy to see that Theorem 2 is true, because otherwise there are arbitrage possibilities at $t = 0$, i.e., static arbitrage possibilities by properly trading at $t = 0$ in the European derivatives $A(\tau)$ and the American option.

Notice, that on the basis of Theorem 2 we do not know yet whether our American option *has* a price process! In order to study the question whether “ A ” allows for a price process that forbids arbitrage we phrase the following definition.

Definition 1. A Q -martingale M (with respect to the given filtration $\{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\}$) is a price candidate for A if the following holds:

- a. $\forall t \in \widehat{T} : M(t, \omega) \geq A(t, \omega), \forall \omega \in \Omega$
- b. $\exists \tau \in \Sigma(0)$ such that $M(\tau) = A(\tau)$.

The economic intuition behind this definition is as follows. First of all “the” price of “ A ” should be at least equal “ A ”. Secondly after the exercise of the option at τ the net proceeds are invested in a selffinancing way such that every time t after τ we still have wealth $M(t)$. Because our underlying market is complete this can be done. Assume in addition that there is a market for “ A ” until time τ . Then it is clear, that there is no arbitrage possible.

We now define:

$$\mathcal{M} := \{M; M \text{ is a price candidate for “} A \text{”}\}.$$

In the sequel we will study in detail this set. The first question is whether \mathcal{M} is empty or not. The answer is given by Theorem 5 reading:

Theorem 5. Let $\hat{\tau} \in \Sigma(0)$ be such that $E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)$. Then there is a Q -martingale M such that:

- a. $\forall t \in \widehat{T} : M(t) \geq A(t)$
- b. $M(\hat{\tau}) = A(\hat{\tau})$.

Later on we will give a proof of this result, as it is based on the following theorem, which is useful on its own.

Theorem 4. Let $\hat{\tau} \in \Sigma(0)$ be such that $E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)$. Then the following is true.

a. Let $\tau^* \in \sigma(0)$ be such that $\tau^* \geq \hat{\tau}$, then:

$$E^Q(P_A(\tau^*) \mid \mathcal{F}_{\tau^*}) \leq A(\hat{\tau}).$$

b. Let $\tau^* \in \Sigma(0)$ be such that $\tau^* \leq \hat{\tau}$, then

$$E^Q(A(\hat{\tau}) \mid \mathcal{F}_{\tau^*}) \geq A(\tau^*).$$

Proof: First we prove part a. To that end we define the set B as follows:

$$B := \{\omega \in \Omega; E^Q(P_A(\tau^*) \mid \mathcal{F}_{\hat{\tau}})(\omega) > A(\hat{\tau})(\omega)\}, \text{ then } B \in \mathcal{F}_{\hat{\tau}}.$$

We now define the random variable τ^0 by:

$$\tau^0 := \chi(B)\tau^* + \chi(B^c)\hat{\tau},$$

where B^c is the complement of B . As $\tau^* \geq \hat{\tau}$ we know that $\mathcal{F}_{\hat{\tau}} \subseteq \mathcal{F}_{\tau^*}$.

For all $t \in \widehat{T}$ we have by construction of τ^0 that:

$$(\tau^0 = t) = (B \cap (\tau^* = t)) \cup (B^c \cap (\hat{\tau} = t)) \in \mathcal{F}_t.$$

Therefore τ^0 is a stopping time. Remember, that $P_A \geq A$ and $P_A(t-1) \geq E^Q(P_A(t) \mid \mathcal{F}_{t-1}), \forall t \in \widehat{T}, t \geq 1$. The last statement implies that $P_A(0) \geq E^Q(P_A(\hat{\tau})) \geq E^Q(A(\hat{\tau})) = P_A(0)$ and hence we have $P_A(\hat{\tau}) = A(\hat{\tau})$. We also have for every $\tau \in \Sigma(0)$ that $P_A(0) \geq E^Q(P_A(\tau))$, and hence we have the following result:

$$P_A(0) = E^Q(P_A(\hat{\tau})) \geq E^Q(P_A(\tau)), \forall \tau \in \Sigma(0).$$

We now consider: $E^Q(P_A(\tau^0)) = E^Q(E^Q(P_A(\tau^0) \mid \mathcal{F}_{\hat{\tau}}))$.

$$\begin{aligned} E^Q(P_A(\tau^0) \mid \mathcal{F}_{\hat{\tau}}) &= E^Q(\chi(B)P_A(\tau^0) \mid \mathcal{F}_{\hat{\tau}}) + E^Q(\chi(B^c)P_A(\tau^0) \mid \mathcal{F}_{\hat{\tau}}) \\ &= \chi(B)E^Q(P_A(\tau^*) \mid \mathcal{F}_{\hat{\tau}}) + \chi(B^c)E^Q(P_A(\hat{\tau}) \mid \mathcal{F}_{\hat{\tau}}) \\ &> P_A(\hat{\tau}), \text{ and hence } E^Q(P_A(\hat{\tau})) > P_A(0), \end{aligned}$$

a contradiction. and therefore part a. is true.

Now we will prove part b. To that end we take $\tau^* \leq \hat{\tau}$ and define

$$B := \{\omega \in \Omega : E^Q(A(\hat{\tau}) \mid \mathcal{F}_{\tau^*})(\omega) < A(\tau^*)(\omega)\}.$$

Obviously $B \in \mathcal{F}_{\tau^*}$. We now define $\tau_0 := \chi(B)\tau^* + \chi(B^c)\hat{\tau}$. For all $t \in \widehat{T}$ we have:

$$(\tau_0 = t) = (B \cap (\tau^* = t)) \cup (B^c \cap (\hat{\tau} = t)) \in \mathcal{F}_t,$$

hence τ_0 is a stopping time. We now calculate $E^Q(A(\tau^0)) = E^Q(E^Q(A(\tau^0) \mid \mathcal{F}_{\tau^*}))$. But

$$E^Q(A(\tau^0) \mid \mathcal{F}_{\tau^*}) = \chi(B)E^Q(A(\tau^*)) + \chi(B^c)E^Q(A(\hat{\tau}) \mid \mathcal{F}_{\tau^*}).$$

Hence $E^Q(A(\tau^0)) > E^Q(A(\hat{\tau}))$, a contradiction we are done with the proof. \square

We now are finally ready to prove Theorem 5!

Proof of Theorem 5: Let $\hat{\tau} \in \Sigma(0)$ be such that

$$E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0).$$

We define the martingale M as follows:

For $0 \leq t \leq \hat{\tau}$ we define $M(t) := E^Q(A(\tau) \mid \mathcal{F}_t)$. When $\eta \in \Sigma(0)$ is an arbitrary stopping time, then $\eta(\text{next}) := (\eta + 1) \wedge T$ is also a stopping time in $\Sigma(0)$. Based on Theorem 4, part a, we know that

$$E^Q(P_A(\hat{\tau}(\text{next})) \mid \mathcal{F}_{\hat{\tau}}) \leq A(\hat{\tau}).$$

Hence there is an $\alpha(\tau) \in m\mathcal{F}_{\tau}$ with $\alpha(\tau) \geq 1$ such that

$$E^Q(\alpha(\tau)P_A(\hat{\tau}(\text{next})) \mid \mathcal{F}_{\hat{\tau}}) = A(\hat{\tau}) = M(\tau).$$

We now define $M(\tau(\text{next})) = \alpha(\tau)P_A(\hat{\tau}(\text{next}))$.

Continuing this way we *construct* a martingale M with the required properties. \square

Some comments are in order now.

1. This is certainly not the shortest proof of the existence of a martingale $M \in \mathcal{M}$. Using the fact P_A is a supermartingale and using the Doob-Meyer decomposition of P_A (an easy result in a discrete-time setting) one really gets a shorter proof.
2. Theorem 4, however, is economically interesting, and leads, intuitively at least, to the knowledge that Theorem 5 must be true. Intuitively Theorem 4 says that stopping at time $\hat{\tau}$ always gives you, in principle, at least as much money as when you exercise at any other moment, you only have to invest in a selffinancing portfolio in a proper way after time $\hat{\tau}$. Of course it is here, that we use the completeness of the underlying market.

The next result gives more information about the set \mathcal{M} .

Theorem 6. Let $M \in \mathcal{M}$ and $N \in \mathcal{M}$ be such that for $\mu \in \Sigma(0)$ and $v \in \Sigma(0)$ the following holds:

$$M(\mu) = A(\mu) \text{ and } N(v) = A(v).$$

Then the following is true:

$$M(\mu) = N(\mu) \text{ and } M(v) = N(v).$$

Proof: By definition of the set \mathcal{M} the following is true: $N(\mu) \geq A(\mu)$ and $M(v) \geq A(v)$. Therefore we have:

$$\begin{aligned} N(0) &= E^Q(N(\mu)) \geq E^Q(A(\mu)) = E^Q(M(\mu)) = M(0) \text{ and} \\ M(0) &= E^Q(M(v)) \geq E^Q(A(v)) = E^Q(N(v)) = N(0). \end{aligned}$$

Now it is easy to see that $M(v) = N(v)$ and $M(\mu) = N(\mu)$. \square

No we will prove the following result:

Theorem 7.

- a. Let $M \in \mathcal{M}$ and $\mu \in \Sigma(0)$ be such that $M(\mu) = A(\mu)$. Then $E^Q(A(\mu)) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)$.
- b. Let $M \in \mathcal{M}$ and let $\hat{\tau} \in \Sigma(0)$ be such that $E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)$, then $M(\hat{\tau}) = A(\hat{\tau})$.

Proof: First we prove a. By definition of \mathcal{M} we have $M(\tau) \geq A(\tau), \forall \tau \in \Sigma(0)$, and M is a martingale, hence $E^Q(M(\mu)) = E^Q(M(\tau)), \forall \tau \in \Sigma(0)$. Therefore we have: $E^Q(A(\mu)) = E^Q(M(\mu)) = E^Q(M(\tau)) \geq E^Q(A(\tau))$, and the proof of part a. is complete.

Now we take $M \in \mathcal{M}$ and $\hat{\tau} \in \Sigma(0)$ such that $E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)$. We know: $M(\hat{\tau}) \geq A(\hat{\tau})$, then $E^Q(M(\hat{\tau})) \geq E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)$. We know that $M(\mu) = A(\mu)$ for some $\mu \in \Sigma(0)$.

We therefore know the following:

$$E^Q(M(\hat{\tau})) = M(0) \geq E^Q(A(\hat{\tau})) \geq E^Q(A(\mu)) = E^Q(M(\mu)) = M(0),$$

and hence (remember that $M(\hat{\tau}) \geq A(\hat{\tau})$) we have $M(\hat{\tau}) = A(\hat{\tau})$, and we are done with the proof. \square

The next result describes a bit more the set of all optimal stopping times.

Theorem 8. Let $\widehat{\Sigma} := \{\hat{\tau} \in \Sigma(0); E^Q(A(\hat{\tau})) \geq E^Q(A(\tau)), \forall \tau \in \Sigma(0)\}$. Then, for some $n \in \mathbf{N}$: $\widehat{\Sigma} = \{\tau_1, \tau_2, \dots, \tau_n\}$. We also have

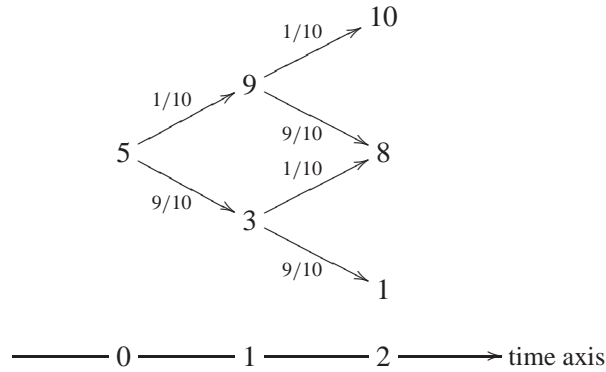
$$\begin{aligned} \tau_{\min} &:= \tau_1 \wedge \tau_2 \dots \wedge \tau_n \in \widehat{\Sigma} \text{ and} \\ \tau_{\max} &:= \tau_1 \vee \tau_2 \dots \vee \tau_n \in \widehat{\Sigma}. \end{aligned}$$

Proof: As Ω is a finite set we trivially have that $\widehat{\Sigma}$ is a finite set. Theorem 7b. implies the existence of a martingale $M \in \mathcal{M}$ such that $M(\tau_i) = A(\tau_i), \forall i \in \{1, 2, \dots, n\}$. Hence we also have: $M(\tau_{\min}) = A(\tau_{\min})$ and $M(\tau_{\max}) = A(\tau_{\max})$. Theorem 7a., together with the fact that τ_{\min} and τ_{\max} are stopping times, implies that $\tau_{\min} \in \widehat{\Sigma}$ and $\tau_{\max} \in \widehat{\Sigma}$, and the proof of this result is finished. \square

Notice that Theorem 8 implies that elements of \mathcal{M} only differ possibly after τ_{\max} . It is now quite reasonable to call τ_{\min} the minimal lifetime of the American option and τ_{\max} the maximal lifetime of the option.

We will end the study of \mathcal{M} and $\widehat{\Sigma}$ with some examples and some concluding remarks, after we have investigated whether \mathcal{M} is a convex set.

Examples By means of the following tree we define our american option A .



To be precise, our time axis is $\widehat{T} = \{0, 1, 2\}$.

Our outcome space Ω has four elements, $\{\omega_1, \omega_2, \omega_3, \omega_4\} = \Omega$. The equivalent martingale measure Q is defined by:

$$Q\{\omega_1, \omega_2\} = \frac{1}{10}, \quad Q\{\omega_3, \omega_4\} = \frac{9}{10}$$

$$Q\{\omega_1\} = \frac{1}{100}, \quad Q\{\omega_2\} = \frac{9}{100}, \quad Q\{\omega_3\} = \frac{9}{100}, \quad Q\{\omega_4\} = \frac{81}{100}.$$

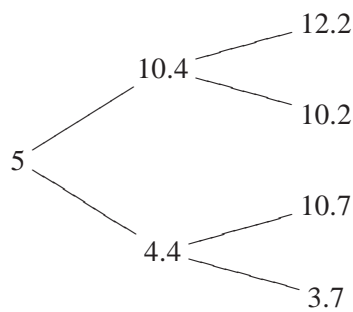
The American option A is defined by:

$$A(0) = 5, \quad A(1, \omega_1) = A(1, \omega_2) = 9, \quad A(1, \omega_3) = A(1, \omega_4) = 3.$$

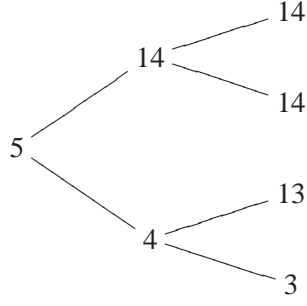
$$A(2, \omega_1) = 10, \quad A(2, \omega_2) = A(2, \omega_3) = 8, \quad A(2, \omega_4) = 1.$$

As filtration we take the natural filtration associated with the stochastic process A .

It is easy to see, that $A = P_A$. The martingale $M_1 = \mathcal{M}$ associated with the Doob-Meyer decomposition of P_A is given by a nonrecombining martingale:



Another martingale $M_2 \in \mathcal{M}$ is given by:



It is clear that $\tau = 0$ is the only optimal stopping time.

Now assume that we replace the number 8 in “A”, at time $t = 2$, by the number 10. The rest of the mapping A is left unchanged. It is easily seen that $\tau = 0$ is still the only optimal stopping time. But . . . once the holder of the option has made a mistake and is at time $t = 1$ at “state” ω_1 , she better waits until $t = 2$ in order to exercise the option!

Although $\widehat{\Sigma}$, the set of optimal stopping times, is a finite set, the set \mathcal{M} , of price candidates for A is not! It is easily seen, however, that \mathcal{M} is a convex set. When we have a closer look at the definition of \mathcal{M} we see that in our example \mathcal{M} is a *polyhedral* convex set.

Theorem. \mathcal{M} is a polyhedral convex set.

Proof: By definition $\mathcal{M} = \{M : \widehat{T} \times \Omega \rightarrow \mathbb{R}; \text{ such that } M \text{ is a martingale with respect to } \{\mathcal{F}_0, \mathcal{F}_1, \dots, \mathcal{F}_T\} \text{ and such that } M \geq A \text{ and } M(\tau) = A(\tau) \text{ for some stopping time } \tau \in \Sigma(0)\}$. Theorem 7b. makes clear that for every $\hat{\tau} \in \Sigma(0)$ with $E^Q(A(\hat{\tau})) \geq E^Q(A(\tau))$, $\forall \tau \in \Sigma(0)$ we have $M(\hat{\tau}) = A(\hat{\tau})$, $\forall M \in \mathcal{M}$. It is now rather obvious that \mathcal{M} is a polyhedral convex set.

Concluding remarks In the previous pages we have walked an unusual route in explaining the basics of the theory of American options. Along the way we gained more insight in the set of optimal stopping times and the set \mathcal{M} of possible price candidates of an American option. In this setting we also gave a precise meaning to the notion of “lifetime of an American option”. One of our examples makes clear that once dead does not imply: never alive again. So in that respect an American option is a strange creature.

Although a bit unusual our treatment is certainly not completely new. Nevertheless we could not find an approach like the one offered above and as we believe that it is useful, at least pedagogically, we decided to write this short note.

As everything we used is quite standard we refrained from a list of references.